

# Hubbard Models as Fusion Products of Free Fermions

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## Abstract

A class of recently introduced  $su(n)$  ‘free-fermion’ models has recently been used to construct generalized Hubbard models. I derive an algebra defining the ‘free-fermion’ models and give new classes of solutions. I then introduce a conjugation matrix and give a new and simple proof of the corresponding decorated Yang-Baxter equation. This provides the algebraic tools required to couple in an integrable way two copies of free-fermion models. Complete integrability of the resulting Hubbard-like models is shown by exhibiting their  $L$  and  $R$  matrices. Local symmetries of the models are discussed. The diagonalization of the free-fermion models is carried out using the algebraic Bethe Ansatz.

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# 1 Introduction

The two-dimensional Hubbard model was introduced to describe the effects of correlation for  $d$ -electrons in transition metals [1]. It was then shown to be relevant to the study of high- $T_c$  superconductivity of cuprate compounds.

The one-dimensional version also has interesting features. The model is integrable [2, 3, 4]. The integrability framework of the model is the quantum inverse scattering method [5, 6, 7]. However, despite sharing many properties with discrete quantum integrable models, the model had a peculiar integrable structure which defined a class of its own. It was therefore natural to look for integrable generalizations. Mapping the fermionic model to a bosonic one with a Jordan-Wigner transformation reveals interesting properties. The local fermionic symmetries become non-local. This has been known for some time but does not seem to have been explored further [31].

Another interesting feature of the 1D Hubbard model and of most interacting one-dimensional systems is their Luttinger liquid behavior [8, 9]. Such a behavior is not restricted to one dimension however [10].

A bosonic  $su(n)$  Hubbard model which contains the usual  $su(2)$  model was recently introduced in [11]. These models were shown to be integrable and to have an extended  $su(n)$  symmetry [12]. The model is built by coupling two copies of the recently discovered  $su(n)$  XX ‘free-fermions’ models [13]. For  $n = 2$  a fermionic formulation exists, but for  $n > 2$  finding an analogous framework remains a tantalizing problem. Strictly speaking, the name free-fermions model does not seem appropriate. It is easy to convince oneself on dimensional grounds that an expression in terms of fermionic operators can only happen for a subclass of models. However for want of a better characterization I shall stick to the foregoing appellation.

Other types of generalizations of the Hubbard model exist. They are mostly of the fermionic type, that is, built from fermionic operators; see for example [14, 15, 16, 17].

In this work I look for new solutions of the Yang-Baxter equation which share the same features as the known XX models. I derive an algebra which unifies the different ‘free-fermions’ representations, and greatly simplifies the calculations. This algebra is reminiscent of the Temperley-Lieb algebra. The former algebra is more restrictive and all the representations found so far are also representations of the Temperley-Lieb algebra. Defining a conjugation operator allows for a simple and new derivation of the decorated Yang-Baxter equation. This equation, introduced by Shastry while studying the usual Hubbard model, is an important algebraic component of the integrability proof for the ‘fusion’ of two models. The ‘fusion’ or coupling of two commuting free-fermions copies is then described, along with an algebraic proof of the integrability of the resulting Hubbard-like models. I then give new representations of the free-fermions algebra and of the conjugation matrix. I discuss symmetry issues related to these models. Diagonalization of the ‘free-fermions’ models using the algebraic Bethe Ansatz method, shows that their spectrum is highly degenerate and simple in a certain sense. Some outstanding issues and possible directions are discussed in the conclusion.

## 2 A new algebra

Let  $E^{\alpha\beta}$  be the  $n \times n$  matrix with a one at row  $\alpha$  and column  $\beta$  and zeros otherwise. Consider the  $\check{R}$ -matrix of the  $su(n)$  XX model [13]:

$$\begin{aligned}\check{R}(\lambda) = & a(\lambda) [E^{nn} \otimes E^{nn} + \sum_{\alpha, \beta < n} E^{\alpha\alpha} \otimes E^{\beta\beta}] \\ & + b(\lambda) \sum_{\alpha < n} (xE^{\alpha n} \otimes E^{n\alpha} + x^{-1}E^{n\alpha} \otimes E^{\alpha n}) \\ & + c(\lambda) \sum_{\alpha < n} (E^{\alpha\alpha} \otimes E^{nn} + E^{nn} \otimes E^{\alpha\alpha})\end{aligned}\tag{1}$$

where  $a(\lambda) = \cos(\lambda)$ ,  $b = \sin(\lambda)$  and  $c(\lambda) = 1$ . The functions  $a$ ,  $b$  and  $c$  satisfy the ‘free-fermion’ condition:  $a^2 + b^2 = c^2$ . For this set of parameters, a Jordan-Wigner transformation turns the  $U = 0$  hamiltonian density for  $su(2)$  into a fermionic expression for free fermions hopping on the lattice.

I now look for  $R$ -matrices having the above form, namely

$$\check{R}(\lambda) = P^{(1)} + P^{(2)} \cos(\lambda) + P^{(3)} \sin(\lambda)\tag{2}$$

and impose the property of regularity

$$\check{R}(0) = \mathbb{I}\tag{3}$$

where  $\mathbb{I}$  is the identity operator. One therefore has  $P^{(1)} + P^{(2)} = \mathbb{I}$ . There is no loss of generality in choosing the proportionality constant to be one since a solution to the Yang-Baxter equation is defined up to a multiplicative function of  $\lambda$ .

Requiring  $\check{R}$  to satisfy the Yang-Baxter equation

$$\check{R}_{12}(\lambda)\check{R}_{23}(\lambda + \mu)\check{R}_{12}(\mu) = \check{R}_{23}(\mu)\check{R}_{12}(\lambda + \mu)\check{R}_{23}(\lambda)\tag{4}$$

yields a finite set of equations. One develops on a set of linearly independent functions and equates the operatorial coefficients. I do not reproduce here all the equations and rather concentrate on the basic ones:

$$P^{(2)} + (P^{(3)})^2 = \alpha \mathbb{I}\tag{5}$$

$$[A + B, [A, B]] = (1 - 3\alpha)(A - B) + A^3 - B^3\tag{6}$$

$$A^4 + (1 - 2\alpha)A^2 = \beta \mathbb{I}\tag{7}$$

where  $A = P_{12}^{(3)}$ ,  $B = P_{23}^{(3)}$  and  $\alpha, \beta$  are two arbitrary complex numbers that arise upon solving equations of the type  $M_{12} = M_{23}$ . Equation (5) and the regularity equation allow to keep  $P^{(3)}$  as the sole unknown operator. All other equations are therefore equations for  $P^{(3)}$ . Equation (6) means that the Reshetikhin criterion is satisfied [18, 19]. This can be seen as an integrability test for spin chains [20]. It seems at first that the large number of constraints on one operator cannot have a solution. But we already know that (1) is a solution for which  $\alpha = 1$  and  $\beta = 0$ . I have looked for the minimal set of equations which solves all the equations involved in the Yang-Baxter equation and have found the following algebra.

Let  $E_i \equiv P_{ii+1}^{(3)}$ , that is,  $E_i$  acts non-trivially on the adjacent spaces  $i, i+1$ . The defining relations of the free-fermions algebra  $\mathcal{A}$  are:

$$\{E_i^2, E_{i\pm 1}\} = E_{i\pm 1} \quad , \quad E_i^3 = E_i \quad (8)$$

$$E_i E_{i\pm 1} E_i = 0 \quad , \quad E_i E_j = E_j E_i \quad \text{for } |i-j| \geq 2 \quad (9)$$

where  $\{A, B\} = AB + BA$ . The fourth equation just expresses the fact that  $E_i$  and  $E_j$  commute when they act non-trivially on disjoint spaces. There does not seem to be solutions of the Yang-Baxter equations for the foregoing  $R$ -Ansatz unless  $\alpha = 1$  and  $\beta = 0$ .

The above algebra is reminiscent of the Temperley-Lieb (TL) algebra [21]. All the solutions given in section 5 can be put, after a ‘gauge transformation’ (a kind of similarity transformation), in a Temperley-Lieb form. Conversely, the set of solutions of the Temperley-Lieb algebra is much larger and most of its solutions do not map to a free-fermions system. Thus the algebra  $\mathcal{A}$  is much more restrictive. More details are given in section 5.

One then checks that  $P^{(1)}$  and  $P^{(2)}$  form a complete set of projectors on the tensor product space  $\mathbb{C}^n \otimes \mathbb{C}^n$ :

$$P^{(1)} + P^{(2)} = \mathbb{I} \quad , \quad (P^{(1)})^2 = P^{(1)} \quad , \quad (P^{(2)})^2 = P^{(2)} \quad , \quad P^{(1)}P^{(2)} = P^{(2)}P^{(1)} = 0 \quad (10)$$

The operator  $P^{(3)}$  is a square root of the operator  $P^{(1)}$ , and  $P^{(2)}P^{(3)} = P^{(3)}P^{(2)} = 0$ .

The above relations imply that the matrix  $\check{R}$  satisfies the unitarity property

$$\check{R}(\lambda)\check{R}(-\lambda) = \mathbb{I} \cos^2 \lambda \quad (11)$$

### 3 The decorated Yang-Baxter equation

The decorated Yang-Baxter equation is an equation similar in form to the Yang-Baxter equation. It was first introduced in [22] as an algebraic relation at the root of the integrability of the  $su(2)$  bosonic Hubbard model. This underwent a first generalization in [11]. I give here the ingredients needed for the existence of the DYBE.

Assume there exists a ‘conjugation’ matrix  $C$  acting on one copy  $\mathbb{C}^n$  such that

$$C^2 = \mathbb{I} \quad , \quad \{C_i, P_{12}^{(3)}\} = 0 \quad , \quad i = 1, 2 \quad (12)$$

$$C_1 P_{12}^{(3)} = P_{12}^{(3)} C_2 \quad , \quad 2(P_{12}^{(3)})^2 = \mathbb{I} - C_1 C_2 \quad (13)$$

where  $C_1 = C \otimes \mathbb{I}$  and  $C_2 = \mathbb{I} \otimes C$ . Then  $C_i$  commutes with  $P_{12}^{(1)}$  and  $P_{12}^{(2)}$ . These relations imply the following, equivalent, conjugation relations for  $\check{R}$  and  $R = \mathcal{P}\check{R}$ , where  $\mathcal{P}$  is the permutation operator on  $\mathbb{C}^n \otimes \mathbb{C}^n$ :

$$C_i \check{R}_{12}(\lambda) = \check{R}_{12}(-\lambda) C_i \quad , \quad i = 1, 2 \quad (14)$$

$$C_i R_{12}(\lambda) = R_{12}(-\lambda) C_j \quad , \quad i, j = 1, 2 \quad , \quad i \neq j \quad (15)$$

Note that only the anticommutator relations were used here. The remaining relations will be needed in section 4.2. For the  $R$ -matrix (1) one has  $C = \sum_{\alpha < n} E^{\alpha\alpha} - E^{nn}$ .

That the  $R$ -matrices satisfy a decorated Yang-Baxter equations (DYBE) is a simple consequence of the conjugation relations and of the Yang-Baxter equation. Consider the following version of the latter equation:

$$\check{R}_{12}(\lambda - \mu) R_{13}(\lambda) R_{23}(\mu) = R_{13}(\mu) R_{23}(\lambda) \check{R}_{12}(\lambda - \mu) \quad (16)$$

One multiplies (16) by  $C_1 C_2$ , commutes them appropriately and removes one  $C$  using  $C^2 = \mathbb{I}$ .<sup>1</sup> After letting  $\mu \rightarrow -\mu$  one obtains the following version of the DYBE

$$\check{R}_{12}(\lambda + \mu) C_1 R_{13}(\lambda) R_{23}(\mu) = R_{13}(\mu) R_{23}(\lambda) C_2 \check{R}_{12}(\lambda + \mu) \quad (17)$$

It is worth noting that, while the YBE is invariant under a ‘gauge transformation’, the DYBE is not. This is due to the arguments of the matrices.

At this point, it is not clear whether the existence of  $C$  follows from that of  $P^{(3)}$ , although it is the case for the models of section 5.

In [11, 12] we saw how to couple two  $su(n)$  XX models à la Shastry. We generalize this procedure to the foregoing models.

## 4 A new kind of fusion

Coupling two  $su(2)$  XX models in an integrable way in the framework of the Quantum Inverse Scattering Method was first done by Shastry [3]. It was then generalized in [11]. The results of the above sections provide all the ingredients required to couple two solutions of the algebra  $\mathcal{A}$ . Thus the derivation of the two following subsections is algebraic and does not depend on a specific representation.

In the context of the quantum inverse scattering method and  $R$ -matrices, the word fusion has a precise meaning. It refers to the construction of higher-dimensional solutions to the Yang-Baxter equation using lower-dimensional solutions. One uses projection operators and reduces the resulting fused-space dimensionality. For instance, fusing two spin- $\frac{1}{2}$   $su(2)$   $R$ -matrices results, after projections, in an  $R$ -matrix for the spin-1  $\times$  spin-1 representation, with dimension  $3^2 \times 3^2$ .

In the kind of fusion described below, one couples two models without a reduction in dimension for the tensor space of the new model; no projection is implemented. The term fusion here then takes a different meaning.

### 4.1 Lax and transfer matrices

The transfer matrix is the generating functional of the infinite set of conserved quantities. The construction given in [11] is still valid here. We consider two commuting copies of the free-fermion models found in the preceding section. Let us stress that the unprimed and primed copies need *not* be of the same type. For instance, the ‘left’ copy can be  $(n_1, n_2)$  while the ‘right’ copy is  $(n'_1, n'_2)$  with  $n$  not necessarily equal to  $n'$  (see section 5 for the definitions and notation). The Hilbert space of the chain is a tensor product of  $\mathbb{C}^{n \cdot n'}$ , the Hilbert space at each site.

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<sup>1</sup>This simple derivation of the DYBE does not appear in [22] and appears to have been overlooked in the literature.

Consider the matrix

$$I_0(h) = \cosh\left(\frac{h}{2}\right) \mathbb{I} + \sinh\left(\frac{h}{2}\right) C_0 C'_0 = \exp\left(\frac{h}{2} C_0 C'_0\right) \quad (18)$$

where  $C$  and  $C'$  are the conjugation matrices of the models being coupled. The second equality follows from  $C^2 = \mathbb{I}$ . The parameter  $h$  is related to the spectral parameter  $\lambda$  by

$$\sinh(2h) = U \sin(2\lambda) \quad (19)$$

where  $U$  characterizes the strength of the coupling. One chooses for  $h(\lambda)$  the principal branch, which vanishes for vanishing  $\lambda$  or  $U$ . Then for  $U = 0$  the monodromy matrix (21) becomes a tensor product of the two, uncoupled, models. The Lax operator at site  $i$  is given by:

$$L_{0i}(\lambda) = I_0(h) R_{0i}(\lambda) R'_{0i}(\lambda) I_0(h) \quad (20)$$

and the monodromy matrix is a product of Lax operators

$$T(\lambda) = L_{0M}(\lambda) \dots L_{01}(\lambda) \quad (21)$$

where  $M$  is the number of sites on the chain. The transfer matrix is the trace of the monodromy matrix over the auxiliary space 0:

$$\tau(\lambda) = \text{Tr}_0 [(L_{0M} \dots L_{01})(\lambda)] \quad (22)$$

The conserved quantities are then given by

$$H_{p+1} = \left( \frac{d^p \ln \tau(\lambda)}{d\lambda^p} \right)_{\lambda=0}, \quad p \geq 0 \quad (23)$$

This completes the ‘fusion’ of the two models.

A proof that  $H_2$  commutes with  $\tau(\lambda)$  was given in [11]. This proof is algebraic and holds also for the models considered here. It yields relation (19). The proof in the following section ensures the complete integrability and yields (19) again.

The derivative of the matrix  $I$  gives the coupling term appearing in (45). Note that the definition involving a logarithm has two benefits. Besides giving the most local operators, it further disentangles the two copies.

The construction of a non-additive  $R$ -matrix intertwining two  $L$ -matrices at different spectral parameters goes through exactly as in [12]. This automatically implies the exact integrability of the models with periodic boundary conditions.

## 4.2 The $R$ -matrix

The  $R$ -matrix which intertwines the Lax operators,

$$\check{R}(\lambda_1, \lambda_2) L(\lambda_1) \otimes L(\lambda_2) = L(\lambda_2) \otimes L(\lambda_1) \check{R}(\lambda_1, \lambda_2), \quad (24)$$

is again given by

$$\begin{aligned} \check{R}(\lambda_1, \lambda_2) = & I_{12}(h_2) I_{34}(h_1) \left( \alpha \check{R}_{13}(\lambda_1 - \lambda_2) \check{R}_{24}(\lambda_1 - \lambda_2) \right. \\ & \left. + \beta \check{R}_{13}(\lambda_1 + \lambda_2) C_1 \check{R}_{24}(\lambda_1 + \lambda_2) C_2 \right) I_{12}(-h_1) I_{34}(-h_2) \end{aligned} \quad (25)$$

This matrix acts on the product of four auxiliary spaces labeled from 1 to 4, and  $\alpha, \beta$  are to be determined. One then requires relation (24) to be satisfied and uses (16) and (17) to derive the following equation:

$$\begin{aligned} & \left( \alpha \check{R}_{13}(\lambda_1 - \lambda_2) \check{R}_{24}(\lambda_1 - \lambda_2) + \beta C_3 \check{R}_{13}(\lambda_1 + \lambda_2) C_4 \check{R}_{24}(\lambda_1 + \lambda_2) \right) I_{12}(2h_1) I_{34}(2h_2) = \\ & I_{12}(2h_2) I_{34}(2h_1) \left( \alpha \check{R}_{13}(\lambda_1 - \lambda_2) \check{R}_{24}(\lambda_1 - \lambda_2) + \beta \check{R}_{13}(\lambda_1 + \lambda_2) C_1 \check{R}_{24}(\lambda_1 + \lambda_2) C_2 \right) \end{aligned}$$

Expanding the exponentials (see (18)) and using (14) results in the cancellation of half the terms on each side. Using (2) and relations (12–13) for all the terms, yield only two equations:

$$\frac{\beta}{\alpha} = \frac{b}{B} \tanh(h_1 + h_2) \quad , \quad \frac{\beta}{\alpha} = \frac{a}{A} \tanh(h_1 - h_2) \quad (26)$$

where  $a = \cos(\lambda_1 - \lambda_2)$ ,  $b = \sin(\lambda_1 - \lambda_2)$ ,  $A = \cos(\lambda_1 + \lambda_2)$  and  $B = \sin(\lambda_1 + \lambda_2)$ . The compatibility equation

$$\frac{\tan(\lambda_1 - \lambda_2)}{\tan(\lambda_1 + \lambda_2)} = \frac{\tanh(h_1 - h_2)}{\tanh(h_1 + h_2)} \quad (27)$$

is satisfied if and only if equation (19) is satisfied for the pairs  $(\lambda_1, h_1)$  and  $(\lambda_2, h_2)$ . One can then factor out  $\alpha = \alpha(\lambda_1, \lambda_2)$  which appears as an arbitrary normalization of the  $R$ -matrix, to obtain:

$$\begin{aligned} \check{R}(\lambda_1, \lambda_2) &= \alpha(\lambda_1, \lambda_2) I_{12}(h_2) I_{34}(h_1) \left( \check{R}_{13}(\lambda_1 - \lambda_2) \check{R}_{24}(\lambda_1 - \lambda_2) + \frac{\sin(\lambda_1 - \lambda_2)}{\sin(\lambda_1 + \lambda_2)} \right. \\ &\quad \times \left. \tanh(h_1 + h_2) \check{R}_{13}(\lambda_1 + \lambda_2) C_1 \check{R}_{24}(\lambda_1 + \lambda_2) C_2 \right) I_{12}(-h_1) I_{34}(-h_2) \quad (28) \end{aligned}$$

The underlying algebraic structure at work here is the one elucidated in sections 2 and 3. We stress here that this proof is rigorous and valid independently of the specific representation for  $P^{(3)}$  and  $C$ . It only involves algebraic properties.

We conclude the integrability proof. The monodromy matrix being a tensor product of  $M$  copies of  $L$  matrices, one has

$$\check{R}(\lambda_1, \lambda_2) T(\lambda_1) \otimes T(\lambda_2) = T(\lambda_2) \otimes T(\lambda_1) \check{R}(\lambda_1, \lambda_2) \quad (29)$$

Taking the trace over the auxiliary spaces, and using the cyclicity property of the trace, one obtains  $[\tau(\lambda_1), \tau(\lambda_2)] = 0$ . We have thus proven that all the conserved charges  $H_p$  mutually commute.

The matrix  $\check{R}$  satisfies the regularity property

$$\check{R}(\lambda_1, \lambda_1) = \alpha(\lambda_1, \lambda_1) \mathbb{I} \quad (30)$$

and the unitarity property:

$$\begin{aligned} \check{R}(\lambda_1, \lambda_2) \check{R}(\lambda_2, \lambda_1) &= \alpha^2(\lambda_1, \lambda_2) \cos^2(\lambda_1 - \lambda_2) \\ &\quad \times \left( \cos^2(\lambda_1 - \lambda_2) - \cos^2(\lambda_1 + \lambda_2) \tanh^2(h_1 - h_2) \right) \mathbb{I} \quad (31) \end{aligned}$$

The intertwiner  $\check{R}$  satisfies a Yang-Baxter relation of its own:

$$\check{R}_{12}(\lambda_2, \lambda_3) \check{R}_{23}(\lambda_1, \lambda_3) \check{R}_{12}(\lambda_1, \lambda_2) = \check{R}_{23}(\lambda_1, \lambda_2) \check{R}_{12}(\lambda_1, \lambda_3) \check{R}_{23}(\lambda_2, \lambda_3) \quad (32)$$

where  $\lambda$  and  $h$  are related through (19). As explained in [12] the direct verification of this relation is tedious, but can be avoided. The proof done for  $su(2)$  in [23] generalizes; this proof is based on Korepanov's tetrahedral Zamolodchikov algebra [24]. All variants of the proof follow from the foregoing algebraic structure. A notable feature of the matrix  $\check{R}(\lambda_1, \lambda_2)$  is its non-additivity property; the  $\lambda$  dependence cannot be reduced to a difference  $(\lambda_1 - \lambda_2)$ . This is the source of the difficulty in verifying (32).

I now give new solutions of the YBE which have the foregoing properties.

## 5 New models

Let  $n, n_1$  and  $n_2$  be three integers such that

$$2 \leq n, \quad 1 \leq n_1 \leq n_2 \leq n-1, \quad n_1 + n_2 = n \quad (33)$$

and  $A, B$  be two disjoint sets whose union is the set of basis states of  $\mathbb{C}^n$ , with  $\text{card}(A) = n_1$  and  $\text{card}(B) = n_2$ . Let

$$P^{(3)} = \sum_{a \in A} \sum_{\beta \in B} \left( x_{a\beta} E^{\beta a} \otimes E^{a\beta} + x_{a\beta}^{-1} E^{a\beta} \otimes E^{\beta a} \right) \quad (34)$$

$$P^{(1)} = (P^{(3)})^2 = \sum_{a \in A} \sum_{\beta \in B} \left( E^{\beta\beta} \otimes E^{aa} + E^{aa} \otimes E^{\beta\beta} \right) \quad (35)$$

$$P^{(2)} = \mathbb{I} - P^{(1)} = \sum_{a, a' \in A} E^{aa} \otimes E^{a'a'} + \sum_{\beta, \beta' \in B} E^{\beta\beta} \otimes E^{\beta'\beta'} \quad (36)$$

The  $n_1, n_2$  parameters  $x_{a\beta}$  are arbitrary complex numbers. Latin indices belong to  $A$  while greek indices belong to  $B$ . These operators satisfy all the defining relations of the algebra  $\mathcal{A}$  and therefore one has an  $R$ -matrix

$$\check{R}(\lambda) = P^{(1)} + P^{(2)} \cos(\lambda) + P^{(3)} \sin(\lambda) \quad (37)$$

which satisfies the YBE. Denote this representation by  $(n_1, n_2)$ .

The conjugation matrix is defined up to an overall sign; it is given by

$$C = \sum_{\beta \in B} E^{\beta\beta} - \sum_{a \in A} E^{aa} \quad (38)$$

and satisfies all the relations of the preceding section. The DYBE therefore holds.

Note that, unless one wants to perform numerical calculations or write down an  $n^2 \times n^2$  matrix representation, one need not specify which states belong to which set, thereby keeping a 'symmetrical' labeling. Note also that the restriction  $n_1 \leq n_2$  is not essential. It just avoids a double counting of distinct models since one has the obvious symmetry  $A \leftrightarrow B$ . The number of models, for a given  $n$ , is equal to the integer part of  $n/2$ :  $\lfloor \frac{n}{2} \rfloor$ .

It is possible to perform a gauge transformation which puts these models in a TL form.<sup>2</sup> One obtains

$$\check{R}^{\text{TL}}(\lambda) = \mathbb{I} \cos(\lambda) + (P^{(3)} + P^{\text{TL}}) \sin(\lambda) \quad (39)$$

$$P^{\text{TL}} = i \sum_{a \in A} \sum_{\beta \in B} \left( E^{\beta\beta} \otimes E^{aa} - E^{aa} \otimes E^{\beta\beta} \right) \quad (40)$$

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<sup>2</sup>The connection between some XX models and their TL formulation was pointed out by Martins [25].



where  $i^2 = -1$ . The operator  $E = P^{(3)} + P^{\text{TL}}$  satisfies the Temperley-Lieb algebra

$$E_i^2 = 0, \quad E_i E_{i\pm 1} E_i = E_i, \quad E_i E_j = E_j E_i \quad \text{for } |i - j| \geq 2 \quad (41)$$

and  $\check{R}^{\text{TL}}(\lambda)$  the Yang-Baxter equation. Let  $M$  be any invertible matrix. A general class of solution of the TL algebra is given by

$$E_{ab,cd} = M_{ab} M_{cd}^{-1}, \quad \text{tr}(M \cdot M^{-1}) = 0 \quad (42)$$

where, on the left-hand side,  $a, b$  ( $c, d$ ) are the row (column) indices. However, only for  $n = 2$  and  $M \propto \text{antidiag}(1, \pm i)$  does one have a free-fermions model. The other  $(n_1, n_2)$  models do not fit in this scheme. This confirms the statement made in section 2.

For  $n_1 = 1$ ,  $n_2 = n - 1$ , and all the  $x_{a\beta}$  equal to each other, one recovers the  $su(n)$  XX models found in [13]. Allowing the twist parameters  $x$  to be unequal amounts to a multiple deformation of these models. We see in the next section that the degree of symmetry depends on the  $x$ 's.

Before diagonalizing the conserved quantities of the free-fermion models we pause to consider their symmetries and their quadratic defining hamiltonians.

## 6 Hamiltonians and symmetries

Periodic boundary conditions are assumed. Consider first the free-fermions models. The quadratic hamiltonian calculated by (23) is given by

$$H_2 = \sum_i P_{ii+1}^{(3)} \quad (43)$$

The cubic hamiltonian is equal to

$$H_3 = \sum_i [P_{ii+1}^{(3)}, P_{i+1i+2}^{(3)}] \quad (44)$$

This is a boosted form of  $H_2$  [26]. The commutation of  $H_2$  and  $H_3$  is a consequence of the Reshetikhin criterion (6). We expect considerations about the boost structure and the explicit form of the conserved quantities to generalize straightforwardly [13].

The quadratic Hubbard-like hamiltonians obtained by fusion are given by

$$H_2 = \sum_i P_{ii+1}^{(3)} + \sum_i P_{ii+1}'^{(3)} + U \sum_i C_i C_i' \quad (45)$$

where primed and unprimed quantities correspond to the two commuting copies. The cubic hamiltonian is not given by a boosted form of  $H_2$  [11, 27].

The hamiltonians  $H_2$  are defined in one dimension but can be evidently defined on any lattice; integrability is lost however. These hamiltonians can be written simply in terms of  $su(n)$  hermitian traceless matrices. For  $|x_{a\beta}| = 1$  and  $U$  real the hamiltonians are hermitian.

Because of the structure of the hamiltonians one expects, at least, the diagonal generators to commute with all the conserved quantities. As seen for the models of [13] the symmetry may be larger. One has the following relations

$$\forall a, b \in A \quad [E_i^{ab}, L_{0i}(\lambda)] = -[E_0^{ab}, L_{0i}(\lambda)] \quad \text{iff } x_{a\beta} = x_{b\beta} \quad \forall \beta \in B \quad (46)$$

$$\forall \alpha, \beta \in B \quad [E_i^{\alpha\beta}, L_{0i}(\lambda)] = -[E_0^{\alpha\beta}, L_{0i}(\lambda)] \quad \text{iff } x_{a\alpha} = x_{a\beta} \quad \forall a \in A \quad (47)$$

where  $L = R$ . The linear magnetic-field operators

$$H_1^{ab} = \sum_i E_i^{ab} \quad a, b \in A \quad (48)$$

$$H_1^{\alpha\beta} = \sum_i E_i^{\alpha\beta} \quad \alpha, \beta \in B \quad (49)$$

commute with the transfer matrix if the parameters  $x_{a\beta}$  satisfy the above constraints for the corresponding indices. One just uses the expression (22), its cyclic structure and the relations (46) and (47). The commutation with  $\tau(\lambda)$  implies the commutation with all the hamiltonians  $H_p$ .

In particular all the diagonal operators are symmetries, without any constraints on the parameters  $x$ . When all the parameters are equal to one parameter, say  $x_{a\beta} = x$ ,  $\forall a \in A$ ,  $\forall \beta \in B$ , the full local symmetry is  $su(n_1) \oplus su(n_2) \oplus u(1)$ . This symmetry is largest (in terms of number of generators) for  $n_1 = 1$  and  $n_2 = n - 1$ , that is for the models of reference [13].

It is straightforward to conclude that the models obtained by fusion inherit the local symmetries of their components. Again, when  $x_{a\alpha}^{(l)} = x^{(l)}$  for the left copy and  $x_{b\beta}^{(r)} = x^{(r)}$  for the right copy, the full local symmetry of the model  $(n_1, n_2) \times (n'_1, n'_2)$  is  $(su(n_1) \oplus su(n_2) \oplus u(1)) \times (su(n'_1) \oplus su(n'_2) \oplus u(1))$ .

One consequence is that one can add magnetic-field terms for each symmetry generators, without spoiling the integrability of the models.

## 7 Algebraic Bethe Ansatz

The diagonalization by Bethe Ansatz of the free-fermions models is very similar to the one for the  $su(n)$  XX models. See [13] for additional details omitted here. I am considering the case  $x_{a\beta} = x$  to avoid unessential complications.

The transfer matrix defined in (22) is the trace over the auxiliary space of the monodromy matrix  $T(\lambda)$ . The latter is an  $n$ -dimensional matrix whose entries are operators acting on the Hilbert space  $C^n \otimes \dots \otimes C^n$ , with a copy for every site. The number of sites is  $M$ . Some elements of the monodromy matrix are used to create Ansätze for the eigenstates. When written in components, equation (29) provides the algebraic relations needed to find the action of the transfer matrix on the states.

We now use the following notation for some elements of the monodromy matrix:

$$S = T_{11}, \quad C_a = T_{1a}, \quad a = 2, \dots, n_1, \quad C_\beta = T_{1\beta}, \quad \beta = n_1 + 1, \dots, n \quad (50)$$

where, as usual, the Latin indices belong to  $A$  and the greek indices to  $B$ . The remaining elements are denoted by  $T_{**}$ . The transfer matrix is given by

$$\tau(\lambda) = S(\lambda) + \sum_{a=2}^{n_1} T_{aa}(\lambda) + \sum_{\beta=n_1+1}^n T_{\beta\beta}(\lambda) \quad (51)$$

It is easy to see that the vector  $||1\rangle \equiv |1\rangle \otimes \dots \otimes |1\rangle$  is an eigenvector of the transfer matrix. The only non-vanishing elements of the monodromy matrix on  $||1\rangle$  are:

$$S(\lambda) ||1\rangle = (\cos(\lambda))^M ||1\rangle, \quad T_{\beta\beta}(\lambda) ||1\rangle = (x^{-1} \sin(\lambda))^M ||1\rangle \quad (52)$$

and the action of *all* the  $C$  operators,  $C_a$  ( $a \neq 1$ ) and  $C_\beta$ .

It turns out that it is still possible to construct Bethe Ansatz eigenvectors using the  $C_\beta$  only, namely:

$$|\lambda_1, \dots, \lambda_p\rangle = \sum_{\alpha_p, \dots, \alpha_1} F^{\alpha_p, \dots, \alpha_1} C_{\alpha_1}(\lambda_1) \dots C_{\alpha_p}(\lambda_p) ||1\rangle \quad (53)$$

where the parameters  $\lambda_i$  and the coefficients  $F$  are to be determined. This Ansatz also vanishes identically if  $p > M$ ; the proof of this fact is given below. Equation (29) gives the following relations:

$$C_\alpha(\lambda)C_\beta(\mu) = C_\rho(\mu)C_\sigma(\lambda)\mathcal{P}_{\sigma\rho, \alpha\beta} \quad (54)$$

$$S(\lambda)C_\beta(\mu) = f(\mu - \lambda)C_\beta(\mu)S(\lambda) + g(\mu - \lambda)C_\beta(\lambda)S(\mu) \quad (55)$$

$$t_{\alpha\beta}(\lambda)C_\gamma(\mu) = f(\lambda - \mu)C_\rho(\mu)t_{\alpha\sigma}(\lambda)\mathcal{P}_{\sigma\rho, \beta\gamma} + g(\lambda - \mu)C_\beta(\lambda)t_{\alpha\gamma}(\mu) \quad (56)$$

$$f(\lambda) = \frac{x \cos \lambda}{\sin \lambda}, \quad g(\lambda) = -\frac{x}{\sin \lambda}, \quad (57)$$

where  $\mathcal{P}$  is the permutation operator for  $\mathbb{C}^{n_2} \otimes \mathbb{C}^{n_2}$ . It is important to notice that the Latin and greek indices do not mix in these relations.

One then applies the transfer matrix on the state  $|\lambda_1, \dots, \lambda_p\rangle$  and with the help of the above relations commutes it through the  $C_\alpha$ 's. The contributions from  $S$  and  $T_{\beta\beta}$  are treated exactly as in [13]. The contributions from  $T_{aa}$  ( $a \neq 1$ ) vanish for  $p < M$  while  $|\lambda_1, \dots, \lambda_p\rangle$  is an eigenvector when  $p = M$ , without any constraint. To see this we need the commutation relations between the Cartan subalgebra generators and the creation operators  $C$ . Using relations (21), (46) and (47), one easily derives

$$[H_1^{11}, C_\beta(\lambda)] = -C_\beta(\lambda) \quad (58)$$

$$[H_1^{aa}, C_\beta(\lambda)] = 0, \quad \forall a \in A - \{1\} \quad (59)$$

Relations (59) imply that the eigenvector Ansatz has no  $a$ -states in it ( $a \neq 1$ ), while relation (58) implies that  $C_{\alpha_1}(\lambda_1) \dots C_{\alpha_p}(\lambda_p) ||1\rangle$  has only  $B$ -states in it when  $p = M$ . This means

$$C_{\alpha_1}(\lambda_1) \dots C_{\alpha_p}(\lambda_p) ||1\rangle \equiv 0 \text{ for } p > M \quad (60)$$

$$T_{aa}(\lambda) C_{\alpha_1}(\lambda_1) \dots C_{\alpha_p}(\lambda_p) ||1\rangle = 0, \quad 0 \leq p \leq M - 1 \text{ and } a \neq 1 \quad (61)$$

$$T_{aa}(\lambda) C_{\alpha_1}(\lambda_1) \dots C_{\alpha_M}(\lambda_M) ||1\rangle = (x \sin(\lambda))^M C_{\alpha_1}(\lambda_1) \dots C_{\alpha_M}(\lambda_M) ||1\rangle, \quad a \neq 1 \quad (62)$$

One then finds the corresponding eigenvalues of  $\tau(\lambda)$

$$\begin{aligned} \Lambda^{((n_1, n_2), M)}(\lambda) &= (\cos(\lambda))^M \prod_{j=1}^p f(\lambda_j - \lambda) + (x^{-1} \sin(\lambda))^M \left( \prod_{j=1}^p f(\lambda - \lambda_j) \right) \Lambda^{(n_2, p)} \\ &\quad + (n_1 - 1)(x \sin(\lambda))^M \delta_{pM} \end{aligned} \quad (63)$$

Here  $\Lambda^{(n_2, p)}$  is an eigenvalue of the unit-shift operator  $\tau^{(n_2, p)}$ , for a chain of  $p$  sites and  $n_2$  possible states at each site; it is constructed out of the permutation operator  $\mathcal{P}$  on  $\mathbb{C}^{n_2} \otimes \mathbb{C}^{n_2}$ . The coefficients  $F^{\alpha_p, \dots, \alpha_1}$  are such  $F$  is an eigenvector of  $\tau^{(n_2, p)}$  for the eigenvalue  $\Lambda^{(n_2, p)}$ . Finally the Bethe Ansatz equations are just

$$(-1)^{p-1} \left( \frac{x \cos(\lambda_j)}{\sin(\lambda_j)} \right)^M = \Lambda^{(n_2, p)}, \quad j = 1, \dots, p \quad (64)$$

The operator  $\tau^{(n_2,p)}$  can be written as a product of disjoint permutation cycles. One also has  $(\tau^{(n_2,p)})^p = \mathbb{I}$ . The eigenvalues  $\Lambda^{(n_2,p)}$  are then, *at most*,  $p^{\text{th}}$  roots of unity and are highly degenerate. The dimensions of the cycles and their multiplicities will depend on both  $p$  and  $n_2$ .

One can perform the above diagonalization procedure over the pseudo-vacuum  $||a\rangle$  ( $a \neq 1$ ). The set of eigenvalues is exactly the same as the one found above and the eigenvectors have the same structure but form a completely distinct set, at least for  $0 \leq p < M$ . This is easily inferred from the action of the Cartan generators. One can also start with any of the vectors  $||\beta\rangle$ ,  $\beta \in B$  and obtain yet other sets of eigenvectors. The superscript  $n_2$  is replaced by  $n_1$  in (63),  $\Lambda^{(n_2,p)}$  and  $\tau^{(n_2,p)}$ . These features reflect a large degeneracy of the spectrum.

Finally, for  $n > 2$ , the eigenvectors  $|\lambda_1, \dots, \lambda_p\rangle$  are generically not eigenvectors of all the magnetic field operators  $H_1^{**}$  [13]. Because the spectrum is degenerate this is not in contradiction with the fact that the magnetic operators commute with the conserved quantities.

We have thus diagonalized the transfer matrix. The nested Bethe Ansatz truncates and the spectrum is trivial in the above sense.

## 8 Conclusion

A new algebra defining bosonic integrable ‘free-fermions’ representations has been derived. These models were shown to have a highly degenerate and ‘simple’ spectrum and to possess local symmetries. Another distinguishing feature of these models is the possibility to couple any two of them in an *integrable* way. The algebraic structure at the root of this ‘fusion’ has been put in a simple algebraic setting, thereby unifying and simplifying all the derivations. New representations were found, generalizing the known XX models [13] and the Hubbard-like models [11].

Finding a multiparametric deformation of  $su(n)$  interpolating between the XXZ models and the models considered here would be interesting [28, 13]. This would provide a quantum group structure which would shed a new light on the models at hand.

Integrable bosonic Bariev chains and their multicopy generalizations have been found [29, 30]. These chains, in their fermionic formulation describe correlated hoppings of electrons on a chain. The bosonic chains are obtained by coupling, in a yet different way, two or more  $su(2)$  XX models. It is therefore likely that one can use the free-fermions models to build new, more general Bariev chains. Martins reported some progress in this direction [25]. The algebra found here is bound to describe the algebraic structure at work for the Bariev chains.

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## References

- [1] M.C. Gutzwiller, Phys. Rev. Lett. **10**, 159 (1963); J. Hubbard, Proc. Roy. Soc. **A276**, 238 (1963).
- [2] E.H. Lieb and F.Y. Wu, Phys. Rev. Lett. **20**, 1445 (1968).
- [3] B.S. Shastry, Phys. Rev. Lett. **56**, 1529 (1986); Phys. Rev. Lett. **56**, 2453 (1986).
- [4] M. Wadati, E. Olmedilla and Y. Akutsu, J. Phys. Soc. Jpn. **56**, 1340 (1987); E. Olmedilla, M. Wadati and Y. Akutsu, J. Phys. Soc. Jpn. **56**, 2298 (1987).
- [5] For reviews on QISM and the Hubbard model see *Proceedings of the Panchgani Winter School* edited by B.S. Shastry, S.S. Jha and V. Singh, Lectures Notes in Physics Vol. 242, (Springer-Verlag, Berlin 1985).
- [6] L.D. Faddeev in *Recent Advances in Field Theory and Statistical Mechanics*, Les Houches (1982); L.A. Takhtajan and L.D. Faddeev, Rus. Math. Sur. **34** (1979) 11.
- [7] H.B. Thacker, Rev. Mod. Phys. **53** (1981) 253; H.J. de Vega, Int. J. Mod. Phys. **A 4** (1989) 2371.
- [8] F.D.M. Haldane, J. Phys. **C 14** (1981) 2585.
- [9] J. Carmelo and A. Avchinikov, J. Phys. **C 3** (1991) 757.
- [10] P.W. Anderson, Phys. Rev. Lett. **64** (1990) 1839.
- [11] Z. Maassarani, *The  $su(n)$  Hubbard model*, LAVAL-PHY-24/97, cond-mat/9709252; accepted for publication in Physics Letters A.
- [12] Z. Maassarani, *Exact integrability of the  $su(n)$  Hubbard model*, LAVAL-PHY-25/97, cond-mat/9710083.
- [13] Z. Maassarani and P. Mathieu, *The  $su(n)$  XX model*, LAVAL-PHY-22/97, cond-mat/9709163; accepted for publication in Nuclear Physics B.
- [14] F. Essler, V. Korepin and K. Schoutens, Phys. Rev. Lett **68** (1992) 2960; Phys. Rev. Lett **70** (1993) 73.
- [15] R.Z. Bariev, A. Klumper, A. Schadschneider and J. Zittartz, Phys. Rev **B 50** (1994) 9676–9679.
- [16] L. Arrachea and A.A. Aligia, Phys. Rev. Lett. **73** (1994) 2240; A.J. Bracken, M.D. Gould, J.R. Links and Y.-Z. Zhang, Phys. Rev. Lett **74** (1995) 2768, cond-mat/9410026.
- [17] A. Kundu, *Quantum integrable family of generalized Hubbard models with twisted Yangian symmetry*, cond-mat/9710033; A.J. Bracken, X.-Y. Ge, Y.-Z. Zhang and H.-Q. Zhou, *Integrable open-boundary conditions for the  $q$ -deformed supersymmetric  $U$  model of strongly correlated electrons*, cond-mat/9710141; D. Arnaudon, *Algebraic approach to  $q$ -deformed supersymmetric variants of the Hubbard model with pair hoppings*, ENSLAPP-A-650/97, physics/9711001; and references therein.

- [18] P.P. Kulish and E.K. Sklyanin, *Integrable Quantum Field Theories*, Springer Lecture Notes in Physics **151** (1982) 61, Eds. J. Hietarinta and C. Montonen.
- [19] T. Kennedy, J. Phys. **A 25** (1992) 2809–2817.
- [20] M. P. Grabowski and P. Mathieu, J. Phys. **A 28**, 4777 (1995).
- [21] H.N.V. Temperley and E.H. Lieb, Proc. Roy. Soc. **A 322** (1971) 251.
- [22] B.S. Shastri, J. Stat. Phys. **50**, (1988) 57–79.
- [23] M. Shiroishi and M. Wadati, J. Phys. Soc. Jpn. **64** (1995) 57.
- [24] I. Korepanov, Commun. Math. Phys. **154** (1993) 85–97, Mod. Phys. Lett. **B 3** (1989) 201–206.
- [25] M.J. Martins, private communication.
- [26] M.G. Tetelman, Sov. Phys. JETP **55** (1981) 306; H.B. Thacker, Physica **D 18** (1986) 348.
- [27] M. P. Grabowski and P. Mathieu, Ann. Phys. **243**, 299 (1995).
- [28] J. Abad and M. Rios, J. Phys. **A 28** (1995) 3319.
- [29] R.Z. Bariev, J. Phys. **A 24** (1991) L549–L553, J. Phys. **A 24** (1991) L919–L923.
- [30] H.-Q. Zhou, Phys. Lett. **A 221** (1996), 104, J. Phys. **A 29** (1996) 5509; H.-Q. Zhou and D.-M. Tong Phys. Lett. **A 232** (1997), 377.
- [31] Work in progress.